

On the Book Thickness of 1-Planar Graphs [★]

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Abstract. In a *book embedding* of a graph G , the vertices of G are placed in order along a straight-line called *spine* of the book, and the edges of G are drawn on a set of half-planes, called the *pages* of the book, such that two edges drawn on a page do not cross each other. The minimum number of pages in which a graph can be embedded is called the *book-thickness* or the *page-number* of the graph. It is known that every planar graph has a book embedding on at most four pages. Here we investigate the book-embeddings of *1-planar* graphs. A graph is *1-planar* if it can be drawn in the plane such that each edge is crossed at most once. We prove that every 1-planar graph has a book embedding on at most 16 pages and every 3-connected 1-planar graph has a book embedding on at most 12 pages. The drawings can be computed in linear time from any given 1-planar embedding of the graph.

1 Introduction

Graph embeddings and linear layouts of graphs play an important role in graph drawing, parallel processing, matrix computation, VLSI design, and permutation sorting. A linear layout prescribes the order in which the vertices are processed and the embedding of the edges reveals structural properties of the given graph. A particular example is a book embedding in which the edges are assigned to pages such that edges in the same page nest and do not cross. Equivalently, the vertices are visited in the linear order and the edges are processed in stacks. The concept of a book embedding of a graph was introduced by Ollmann [14] and by Kainen [12] and can be formalized as follows. A k -page book embedding of a graph $G = (V, E)$ is defined by a linear order of the vertices of G and a partition of the edges into k sets E_1, \dots, E_k , so that the vertices of G are placed on a line in the given order and edges in E_i are drawn on page i (typically with circular arcs), so that no two edges on the same page cross. The book thickness of the graph G is the smallest number of pages needed, also known as stack number or page number.

The book thickness of planar graphs has been studied for over 40 years. Bernhart and Kainen [3] characterized the graphs with book thickness one as the outerplanar graphs and the graphs with book thickness two as the sub-Hamiltonian planar graphs. Deciding whether a general planar graph has book thickness two is NP-hard [8]. It is known that planar graphs require 3 pages and a series of improvements brought down

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the upper bound from 9 [6], to 7 [11], and 4 [19]. Although in an earlier version of his 1989 paper Yannakakis in 1986 [18] claimed 4 pages are necessary, and later Dujmovic and Wood in 2007 [9] also conjectured the same lower bound, there is still no conclusive evidence that this is indeed the case.

More recently there has been a greater interest in studying non-planar graphs which extend planar graphs by restrictions on crossings. A particular example are *1-planar graphs* which can be drawn in the plane with at most one crossing per edge. Such graphs were first defined by Ringel in the context of simultaneously drawing a planar graph and its dual [16]. In many respects, 1-planar graphs generalize planar graphs. There are 1-planar embeddings as witnesses of 1-planarity, in which the crossings are treated as special vertices of degree four, and which then result in planarizations. Like n -vertex planar graphs which have at most $3n - 6$ edges, n -vertex 1-planar graphs have at most $4n - 8$ edges [15]. Both planar and 1-planar 3-connected graphs admit straight-line drawings in $O(n^2)$ area (with the exception of one edge in the outer face for the densest 1-planar graphs) [1]. However, there is a major difference in the complexity of the recognition of planar and 1-planar graphs, which can be done in linear time for planar graphs while it is *NP*-hard for 1-planar graphs [10,13]. On the other hand, there is a cubic time recognition algorithm for hole-free map graphs [7], which for 3-connected graphs coincide with *planar-maximal* 1-planar graphs (i.e., where no edge can be added without creating more crossing).

In this paper, we address the problem of book embedding of 1-planar graphs. Recently Bekos *et al.* [2] gave a constant upper bound of 39 on the book thickness of 1-planar graphs. Here we prove that 1-planar graphs have book thickness at most 16 and 3-connected 1-planar graphs have book thickness at most 12. If the planar skeleton is Hamiltonian, then four pages suffice, and we have found 1-planar graphs which need four pages.

2 Preliminaries

A *drawing* of a graph G is a mapping of G into the plane such that vertices are mapped to distinct points and edges are Jordan arcs between their endpoints. A drawing is *planar* if the edges do not cross and it is *1-planar* if each edge is crossed at most once. Hence in a 1-planar drawing the crossing edges come in pairs. For example, K_5 and K_6 are 1-planar graphs. An *embedding* of a graph is planar (resp. 1-planar) if it admits a planar (resp. 1-planar) drawing. An embedding specifies the *faces*, which are topologically connected regions. The unbounded face is the *outer face*. Accordingly, a *1-planar embedding* $\mathcal{E}(G)$ specifies the faces in a 1-planar drawing of G including the outer face. A 1-planar embedding is a witness for 1-planarity. In particular, $\mathcal{E}(G)$ describes the pairs of crossing edges, the faces where the edges cross, and the *planar* edges.

Augment a given 1-planar embedding $\mathcal{E}(G)$ by adding as many edges to $\mathcal{E}(G)$ as possible so that G remains a simple graph and the newly added edges are planar in $\mathcal{E}(G)$. We call such an embedding a *planar-maximal* embedding of G and the operation *planar-maximal augmentation*. Then each pair of crossing edges is augmented to a K_4 . The *planar skeleton* $\mathcal{P}(\mathcal{E}(G))$ consists of the planar edges of a planar-maximal augmentation. It is a planar embedded graph, since all pairs of crossing edges are omitted.

Note that the planar augmentation and the planar skeleton are defined for an embedding, not for a graph.

The *normal form* for an embedded 3-connected 1-planar graph $\mathcal{E}(G)$ is obtained by first adding the four planar edges to form a K_4 for each pair of crossing edges while routing them close to the crossing edges and then removing old duplicate edges if necessary. Such an embedding of a 3-connected 1-planar graph is a normal embedding of it. A *normal planar-maximal augmentation* for an embedded 3-connected 1-planar graph is obtained by first finding a normal form of the embedding and then by a planar-maximal augmentation.

Given a 1-planar embedding $\mathcal{E}(G)$, the normal planar-maximal augmentation of $\mathcal{E}(G)$ can be computed in linear time [1]. We say that an embedded 3-connected 1-planar graph is a *normal planar maximal* 1-planar graph if a normal planar maximal augmentation of the graph yields the same graph. In a 3-connected normal planar-maximal 1-planar graph, each pair of crossing edges (a, c) and (b, d) crosses each other either inside or outside the boundary of the quadrangle $abcd$ of the planar edges, and these define the so-called *augmented X-* and *augmented B-configurations* [1].

For a 3-connected 1-planar graph G , Alam *et al.* [1] proved the following:

Lemma 1. [1] *Let G be a 3-connected 1-planar graph with a 1-planar embedding $\mathcal{E}(G)$. Then the normal planar-maximal augmentation of $\mathcal{E}(G)$ gives a planar-maximal 1-planar embedding $\mathcal{E}(G^*)$ of a supergraph G^* of G so that $\mathcal{E}(G^*)$ contains at most one augmented B-configuration in the outer face and each augmented X-configuration in $\mathcal{E}(G^*)$ contains no vertex inside its skeleton.*

3 Book Embeddings of 3-Connected 1-Planar Graphs

If a graph can be embedded in a given number of pages, the same is true for its subgraphs. Given an embedded 3-connected 1-planar graph G , we therefore assume that G is a normal planar maximal 1-planar graph. Lemma 1 implies that the planar skeleton of a normal planar maximal 3-connected 1-planar graph G contains only triangular and quadrangular faces. Furthermore if we remove exactly one crossing edge (arbitrarily) from each pair of crossing edges in G , then the resulting graph is a maximal planar graph.

Our algorithm uses the a “peeling technique” similar to Yannakakis [19] and iteratively removes the vertices on the outer cycle of the planar skeleton $\mathcal{P}(\mathcal{E}(G))$ of G . This partitions the vertices of G into *levels* according to their “distance” from the outer face of the planar skeleton $\mathcal{P}(\mathcal{E}(G))$. Vertices on the outer face of $\mathcal{P}(\mathcal{E}(G))$ are at level 0. Deleting these vertices from $\mathcal{P}(\mathcal{E}(G))$ yields the level 1 graph; the vertices that lie now on the outer face are at level 1. In general, the level t graph is obtained by deleting all vertices at levels less than t ; the vertices that lie on the outer face of this graph are at level t . The edges of G (including the crossing edges) are partitioned into *level edges* at level i , edges that connect vertices at the same level i , and *binding edges*, edges that connect vertices at different levels. The fact that a level i vertex is not on the outer face after deleting the first $i - 2$ levels implies that every level i vertex lies in the interior of some cycle composed of level $i - 1$ vertices. This means in particular that a level i

vertex cannot be adjacent to a level j vertex with $j < i - 1$ and binding edges connect only consecutive levels.

Similar to Yannakakis [19] we first place level 0 vertices in the clockwise order (cw-order) as they appear on the outer cycle, assigning the edges on the outer cycle on the same page. Then we place the level 1 vertices and assign the following edges to some pages: (i) the level edges of each cycle on the outer boundary of the level 1 graph (ii) the binding edges between levels 0 and 1 (iii) the crossing edges either at level 0 or binding between level 0 and 1. Level 1 vertices are placed in such a way that the vertices on each level 1 cycle are in the counterclockwise order (ccw-order) around the cycle. Now the rest of the graph is in the interior of level 1 cycles. The algorithm takes each level 1 cycle in turn and lays out its interior in a similar way.

We therefore next consider a 2-level subgraph H of G defined as follows. The vertices of H are the vertices on a level i cycle C_i and all the level $i + 1$ vertices V_{i+1} interior to C_i . The edges of H are all the planar and crossing edges inside the region between C_i and the outer boundaries of all the level $i + 1$ components inside C_i (including the edges on C_i and the level $i + 1$ boundaries). Fig. 1 shows a 2-level subgraph inside a cycle $C_i = AB \dots Z$. We denote this 2-level subgraph of H inside C_i as $H(C_i)$. We assume that C_i has already been embedded where the vertices of C_i are placed in the cw (or ccw, resp.) order around C_i . We then extend this embedding to a book embedding of $H(C_i)$, by placing the remaining vertices of $H(C_i)$ and assign the remaining edges of $H(C_i)$ to seven pages. The book embedding of G is obtained by repeatedly computing the book embeddings of $H(C_i)$ and reusing the same seven pages for all odd (even) i .

3.1 Drawing 2-Level Subgraphs

In this section we prove the following lemma.

Lemma 2. *Let $H(C_i)$ be a 2-level subgraph of G inside a level i cycle C_i . Then there exists a book embedding Γ of $H(C_i)$ on seven pages where the vertices of C_i are placed in the cw (or ccw) order around C_i .*

We give a construction of a book embedding where the vertices of C_i are placed in the cw-order (for ccw-order we flip the embedding of $H(C_i)$). Let v_1, \dots, v_t be the vertices of C_i in the cw-order around C_i . For the remaining part of this section, we call the vertices on C_i as the *outer vertices* and the level $i + 1$ vertices of $H(C_i)$ as the *inner vertices*. We first obtain a planar graph H' from $H(C_i)$ by removing exactly one edge from each pair $\langle (a, b), (c, d) \rangle$ of crossing edges. Let X be the set of crossing edges that we remove. From each crossing edge pair $\langle (a, b), (c, d) \rangle$, we take one edge to be in X as follows; see Fig. 1.

Case S1. If both (a, b) , (c, d) are level edges at level i , then we take the edge adjacent to the vertex farthest from v_1 in cw-order on C_i to be in X . In particular if the two level i edges forming the crossing pair are (v_p, v_r) , (v_q, v_s) with $p < q < r < s$, then we take the edge (v_q, v_s) to X ; for example we take the edge (C, E) to X in Fig. 1.

Case S2. If both (a, b) , (c, d) are binding edges, then we again choose the edge adjacent to the vertex farthest from v_1 in cw-order on C_i to be in X . In particular, if the two

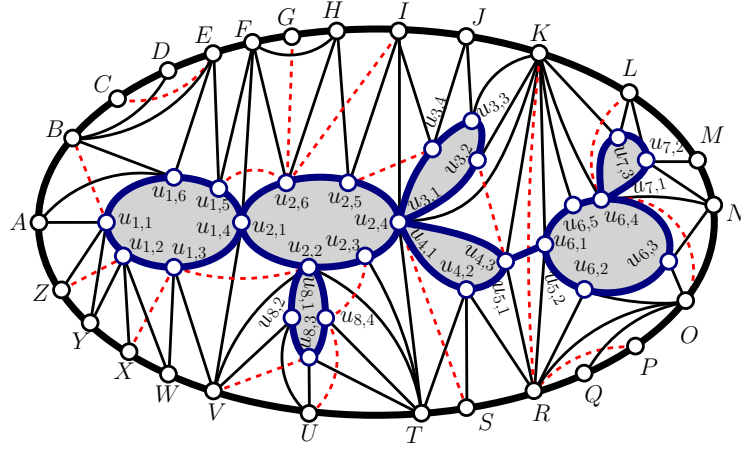


Fig. 1. A 2-level subgraph $H(C_i)$ of G inside the level- i cycle $C_i = AB \dots Z$, which is drawn with thick black edges. The outer boundary of the level $i + 1$ component is drawn with thick blue edges. The red dashed edges are the crossing edges taken in the set X .

binding edges forming the crossing pair are (v_p, u) , (v_q, u') , where $p < q$ and u, u' are level $i + 1$ vertices, then we take the edge (v_q, u') to be in X ; for example we take the edge $(I, u_{2,6})$ to X in Fig. 1.

Case S3. If one of (a, b) , (c, d) is a level edge at level i , and the other is a binding edge, then we choose the binding edge to be in X ; for example we take the edge $(G, u_{2,6})$ to X in Fig. 1.

Case S4. If one of (a, b) , (c, d) is a level edge at level $i + 1$, and the other is a binding edge, then we choose the level edge at level $i + 1$ to be in X ; for example we take the edge $(u_{1,5}, u_{2,6})$ to X in Fig. 1.

Case S5. If one of (a, b) , (c, d) is a level edge at level i , and the other is at level $i + 1$, then we choose the level edge at level i to be in X ; for example we take the edge (K, R) to X in Fig. 1.

Note that due to the construction of $H(C_i)$, the pair of crossing edges cannot both be level edges at level $i + 1$. Thus the above cases account for all possible pairs.

We then use the algorithm by Yannakakis [19] to obtain a book embedding of H' on three pages, and we add the crossing edge from X on four additional pages so that no two edges assigned to the same page cross each other on that page. Before we describe how to add the crossing edges, we describe the 3-page embedding of H' . Denote the three pages as p_1 , p_2 and p_3 , and denote the four additional pages for the crossing edges as c_1 , c_2 , c_3 and c_4 . Denote by D the subgraph of $H(C_i)$ induced by the vertices at level $i + 1$. Assume without loss of generality that D induces a connected graph, since otherwise each connected component of D would be inside a different cycle induced by the vertices of C_i and these can be handled separately. By construction then, each biconnected block of D is a simple cycle (i.e., D is a *cactus graph*). Let B_1, \dots, B_s be these blocks of D and let \mathcal{T} be the block-cut tree for D . We now show how we place

all the level $i + 1$ vertices and assign the edges in H' and in X to the seven pages $p_1, p_2, p_3, c_1, c_2, c_3$ and c_4 .

Placement of Vertices We say that a vertex u *sees* an edge (v, w) if uvw forms a triangular face in H' . We say that an outer vertex *sees* a block B_j of D if it sees an edge of the block. Consider the triangular inner face containing the edge (v_1, v_t) of C_i . The third node of this face $u_{1,1}$ is called the first inner node and assume the block B_1 containing $u_{1,1}$ is the *first block*³. Then consider the block-cut tree \mathcal{T} as a rooted tree by taking B_1 as its root.

For each block B_j of D , define the *leader* of B_j to be the first vertex of B_j in any path from $u_{1,1}$ to any vertex of B_j . Thus, the leader of the root block B_1 is $u_{1,1}$; for any other block B_j , the leader of B_j is the common vertex between B_j and its parent in \mathcal{T} . Although an inner vertex of H' (in particular a cutpoint of D) may belong to more than one block, we assign each inner vertex u to a unique block by assigning it to the *highest* (i.e., closest to the root) block in the tree \mathcal{T} that contains it. Thus, the root block B_1 of \mathcal{T} is assigned all its vertices; each remaining block is assigned all its vertices except its leader. The *dominator* of a block B_j is the first outer vertex (in the order v_1, \dots, v_t) adjacent to a vertex assigned to B_j .

We first place the outer vertices in the order v_1, \dots, v_t in Γ . Next we place the inner vertices in between these outer vertices using the vertex placement order in [19], which we describe here. The inner vertices assigned to each block B_j are placed right after the outer node v_k that dominates B_j (i.e., between v_k and v_{k+1}). If a unique block B_j is dominated by v_k , then its vertices are placed between v_k and v_{k+1} in the ccw-order around its boundary. If more than one block has a common dominator v_k , this set S of blocks forms a directed path in \mathcal{T} . Vertices in these blocks are placed between v_k and v_{k+1} using one of two methods. In the *nested method*, the vertices are placed in the order they are first encountered while traversing the boundary of the subgraph induced by the blocks in S in ccw-order, starting with the leader of the highest block in S . In the *consecutive method*, the vertices assigned to each block are placed consecutively in ccw-order around its boundary; the blocks are ordered one after the other in top-down order of \mathcal{T} : first the vertices assigned to the highest block, then the ones assigned to its child, and so on. For the following description, assume that we follow the consecutive method, (the algorithm is analogous with the nested method). We thus obtain the ordering of the vertices of H' for the book embedding Γ ; see Fig. 2.

Assigning Edges in H' to Pages Next we assign the edges of $H(C_i)$ to the seven pages. For a vertex v of $H(C_i)$, let $\Gamma(v)$ denote its rank in the ordering of Γ . Consider two edge $(a, b), (c, d)$ of $H(C_i)$ with $\Gamma(a) < \Gamma(b)$ and $\Gamma(c) < \Gamma(d)$. We say that there is a conflict between these two edges in Γ if $\Gamma(a) < \Gamma(c) < \Gamma(b) < \Gamma(d)$ or $\Gamma(c) < \Gamma(a) < \Gamma(d) < \Gamma(b)$. We now assign the edges of $H(C_i)$ on seven pages such that there is no conflict between any two edges assigned to the same page.

First we assign the edges of H' to the three pages p_1, p_2 and p_3 . In order to see this assignment of edges to pages, consider H' as a directed (acyclic) graph by taking

³ Assume $u_{1,1}$ is in a unique block; otherwise take as B_1 a block that has $u_{1,1}$ and is seen by v_1 .

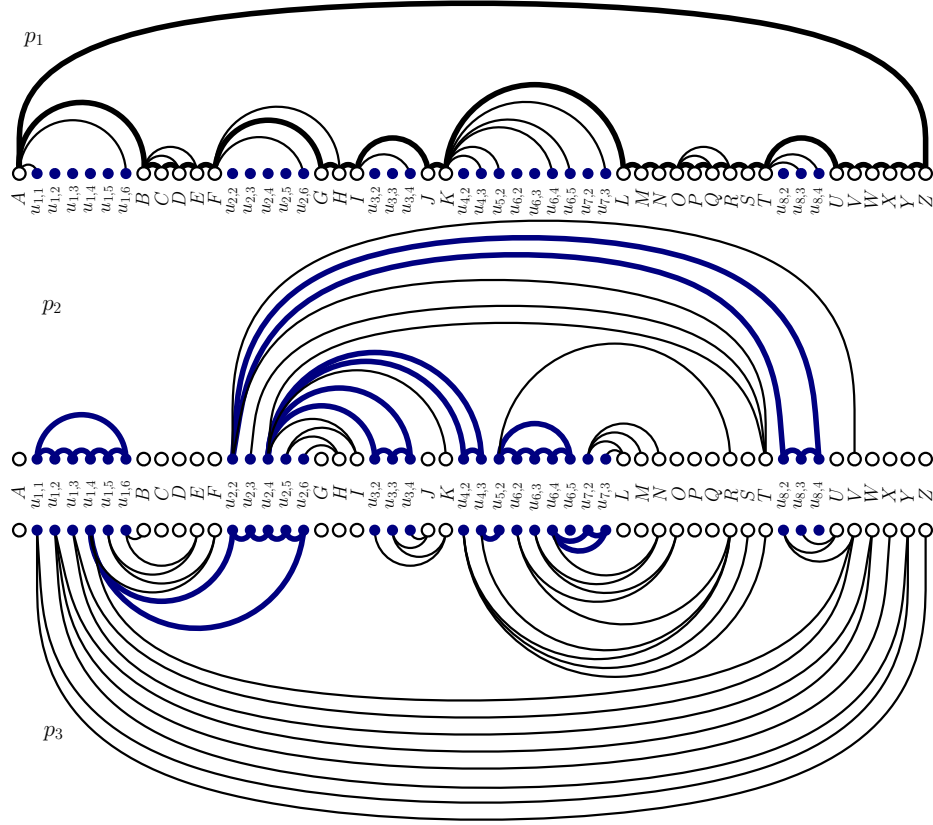


Fig. 2. Book embedding of the planar edges in H' for the 2-level graph $H(C_i)$ in Fig. 1 on the three pages p_1, p_2, p_3 .

the following orientation of edges. The level edges (v_p, v_q) at level i , with $p < q$ are oriented from v_p to v_q (including the edge (v_1, v_t) , which is oriented from v_1 to v_t). On the other hand, each inner cycle is traversed in ccw-order, starting from its leader and the edges are oriented accordingly, with the exception of the final edge, which is oriented away from the leader. Each binding edge is oriented from the inner vertex to the outer vertex. The orientation of edges along with the placement of the vertices in Γ partitions all the edges of H' in two types: *forward* edges have sources placed before their sinks in Γ (the edge orientation is forward); the remaining edges are *backward* edges (the edge orientation is backward).

Consider an assignment of the blocks of D to the pages p_2 or p_3 . The root block is assigned to p_2 . In the nested method, for each non-root block B_i , if B_i has a different dominator than its parent then it is assigned to the opposite page (p_2 or p_3) than that of its parent, otherwise it is assigned to the same page as its parent. In the consecutive method each non-root block B_i is assigned a different page (p_2 or p_3) than that of its

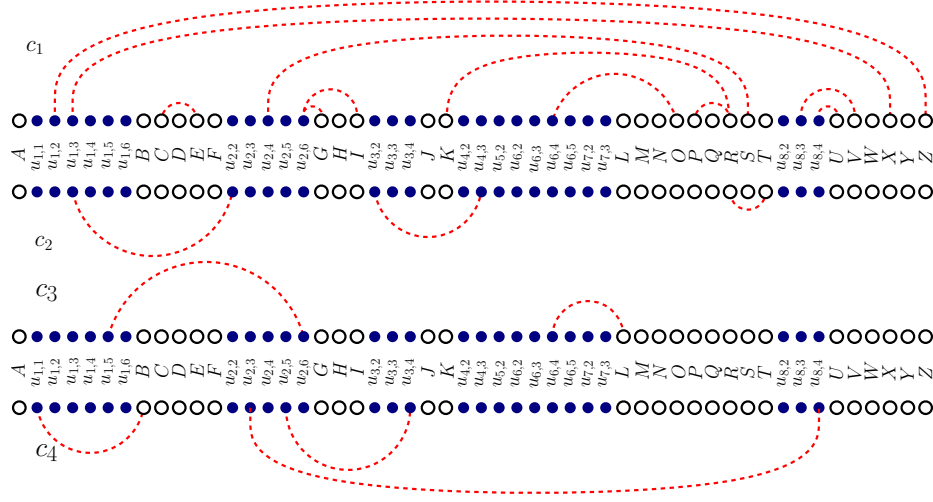


Fig. 3. Book embedding of the crossing edges in X for the 2-level graph $H(C_i)$ in Fig. 1 on the four pages c_1, c_2, c_3, c_4 .

parent. Again we use the consecutive method for illustration. We assign all the edges of H' to the three pages p_1, p_2, p_3 as follows; also see Fig. 2.

- The edges of C_i and all the backward binding edges of H' is assigned to page p_1 . (These are the only edges of H' assigned to p_1 .)
- For each block B_j the level edges in B_j are assigned to the page that the block itself is assigned to.
- For each forward binding edge $e = (u, v)$, where v is on C_i and u is on some block B_j , edge e is assigned to page p_2 or p_3 , opposite to the one assigned to block B_j .

This assignment of edges of H' creates no edge conflicts in any of the three pages [19].

Assigning Edges in X to Pages We now assign the edges in X to the four pages c_1, c_2, c_3 and c_4 . We consider the following cases of a crossing edge (a, b) in X .

Case D1: (a, b) is a binding edge. A binding edge in X is called *forbidden* for some block B_j if it is between two vertices d and v_{k+1} , where d is the leader of B_j , v_k is the dominator of B_j , v_{k+1} is the outer vertex just after v_k and v_k is not the dominator of any child block of B_j in \mathcal{T} . We assign a binding edge (a, b) to page c_1 if it is not forbidden for some block; see Fig. 3; otherwise we assign it to either page c_3 or page c_4 in Case D4.

Case D2: (a, b) is a level i edge. In this case we assign (a, b) to page c_1 ; see Fig. 3.

Case D3: (a, b) is a level $i + 1$ edge. In this case (a, b) is crossed by a binding edge (c, d) , where one vertex (say c) is an outer vertex, and the other vertex (say d) is a cutvertex in D . The four vertices a, b, c, d form a K_4 in $H(C_i)$ with skeleton $acbd$ whose interior is vertex-empty. Let B_j and $B_{j'}$ be the two blocks of D containing a

and b , respectively, with the common vertex d . Then, either one of B_j and $B_{j'}$ is the parent of the other in \mathcal{T} , where d is the leader for the child block, or both B_j and $B_{j'}$ are the children of a common parent block in \mathcal{T} and d is the leader for both of them. In either case, assume without loss of generality that the dominator of B_j comes before the dominator of $B_{j'}$ in the cw-order around C_i (i.e., the dominator of B_j is placed before that of $B_{j'}$ in Γ). This implies that the vertices of B_j are all placed before the vertices of $B_{j'}$ except for its leader. Since b is adjacent to the leader d in $B_{j'}$, b is either the first or the last vertex of $B_{j'}$ (except for its leader) in Γ . We call the edge (a, b) , the *first* (resp. *last*) crossing edge for the block $B_{j'}$. Note that if (a, b) is the last crossing edge for $B_{j'}$, then c is the dominator for $B_{j'}$. We assign a crossing edge in X at level $i + 1$ to page c_2 if it is the first crossing edge for some block $B_{j'}$; see Fig. 3. The last crossing edges of the blocks are assigned to either page c_3 or page c_4 in Case D4.

Case D4: the other case: (a, b) is a forbidden binding edge for some block or the last crossing edge for some block. Since the edges in X do not cross each other, for each block B_j , there is at most one edge, which is either a last crossing edge or a forbidden binding edge for B_j . These edges are assigned to page c_3 or c_4 as follows. Consider the rooted block-cut tree \mathcal{T} for the blocks, rooted at B_1 . For each block at the even (resp. odd) level of \mathcal{T} , we assign its forbidden binding edge or last crossing edge (if any) to page c_3 (resp. c_4); see Fig. 3.

We now prove Lemma 2 by showing that for any of the seven pages, there is no conflict between the edges assigned to it. This follows directly from [19] for the three pages p_1 , p_2 and p_3 , since the edges assigned to these three pages forms the planar graph H' and the order of the vertices and the edge assignment on these three pages for H' is exactly the same as in [19]. For the edges assigned to the remaining four pages c_1 , c_2 , c_3 , c_4 , we have the following Lemmas.

Lemma 3. *There is no conflict between edges assigned to page c_1 .*

Proof. The edges assigned to c_1 are the level i edges and the binding edges in X , not incident to the leader of any block. We show that no two of them create a conflict. Since the vertices of C_i are placed in circular (clockwise) order of its boundary, and no two edges in X crosses each other in the embedding $H(C_i)$ (only one edge from each crossing pair is taken), no two level i edges in X are in conflict with each other. Therefore it is sufficient to show that no binding edge in X is in conflict with any other binding edge or level i edge in X .

Consider a binding edge (x, v_x) assigned to page c_1 , where v_x is an outer vertex and x is an inner vertex; see Fig. 4. Let x is assigned to the block B_j . Let v_k be the dominator of B_j and d be the leader of B_j . Also consider a path P from the first inner vertex $u_{1,1}$ to d in the planar skeleton of $H(C_i)$ (the trivial path if $j = 1$). The block B_j , the two edges (x, v_x) and (d, v_k) , along with the path P and the two edges $(u_{1,1}, v_1)$, $(u_{1,1}, v_t)$ partitions the interior of C_i in the following parts: (i) the interior of

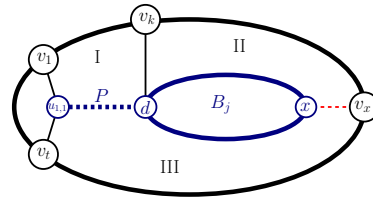


Fig. 4. Illustrations for the proof of Lemma 3.

B_j , (ii) the interior of the triangle $(u_{1,1}, v_1, v_t)$ and (iii) the three regions marked by I, II and III in Fig. 4. Since the path P and the boundary of B_j belongs to the planar skeleton of $H(C_i)$ and since the edge (x, v_x) is a crossing edge, each edge assigned to page c_1 is embedded in the interior of one of the three regions I, II or III.

All the level i vertices in region I are placed on or before v_k in Γ . Since v_k is the dominator of B_j , it is placed before any vertex assigned to B_j , including x . Thus any level i edge in X lying in region I has both their end-vertices placed before both x and v_x , and hence does not create a conflict with (x, v_x) . On the other hand, all level $i + 1$ vertices y in region I including the ones on P are also placed before x . Indeed if $B_{j'}$ is the block to which y is assigned to, then either $B_{j'}$ is dominated by an outer vertex placed before v_k , or $B_{j'}$ is dominated by v_k , but its vertices are placed before those of B_j , following the consecutive (or the nested) method of placement. Thus both end-vertices of any binding edge in X lying in region I are also placed before x and v_x and hence does not create any conflict with (x, v_x) .

Again all the level i vertices in region II except for v_k are placed after x and before v_x . Similarly all the level $i + 1$ vertices in region II except for d are placed on or after x and before v_x . Due to the way, we select the edges in X , no binding edge or level i edge in X lying in region II is incident to v_k . Furthermore no binding edge incident to d are assigned to page c_1 . Thus all the binding edges and level i edges assigned to page c_1 have end-vertices placed between x and v_x ; hence they create no conflict with (x, v_x) .

All the level i vertices in region III are placed on or after v_x . Thus all level i edges in X lying in region III have both their end vertices placed after both x and v_x , and hence they create no conflict with (x, v_x) . On the other hand, the level $i + 1$ vertices on P or on the boundary of B_j lying region III are placed before x and the binding edge incident to them does not create conflict with (x, v_x) . Finally all the level $i + 1$ blocks strictly in region III are dominated by the vertices placed on or after v_x . Indeed, the only possible planar edge crossing the region boundary would have been incident to the level i vertex v_{x-1} just before v_x , and it would have crossed the edge (x, v_x) . However in that case, the other end vertex of such an edge would have been on a block dominated by v_{x-1} and x would have been its leader, which is a contradiction since the edge (x, v_x) is assigned to page c_1 . Thus all the binding edges in region III incident to some level $i + 1$ vertex neither on P nor B_j , have both the end-vertices placed after x and v_x , and hence they do not create conflict with (x, v_x) . \square

For a planar Hamiltonian graph, the order of the vertices from a Hamiltonian cycle induces a 2-page book embedding [3]. Furthermore if the graph is outerplanar, then this order of the vertices on the outer cycle induces a 1-page book embedding. We use these two facts to show that there is no conflict on the pages c_2 , c_3 and c_4 .

Lemma 4. *There is no conflict between edges assigned to the pages c_2 , c_3 and c_4 .*

Proof. Consider a cycle C defined by the vertex order in Γ ; i.e., the vertices of C are all the vertices of $H(C_i)$, and for each consecutive vertex in Γ , there is an edge in C , along with an edge between the first and the last vertex on Γ . We show that all the edges assigned to page c_2 along with this cycle C forms an outerplanar graph with C as the outer cycle. We also show that all the edges assigned to pages c_3 , c_4 , along with C forms a planar graph with the Hamiltonian cycle C . The claim thus follows.

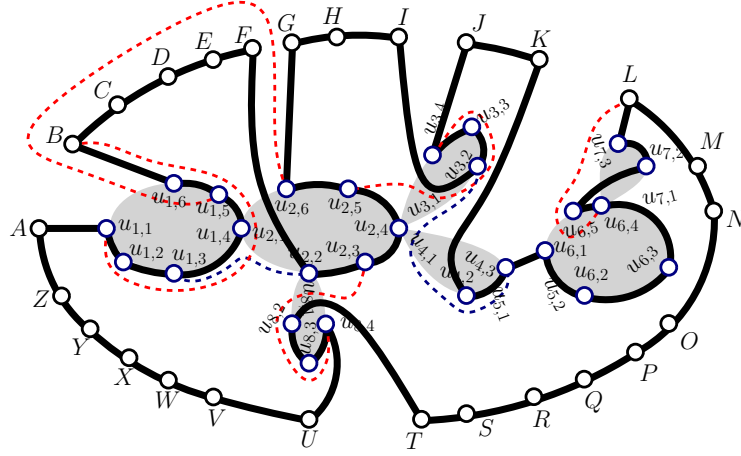


Fig. 5. Construction of the Hamiltonian cycle from Γ (thick black edges). The blue dotted edges are the first crossing edges of the blocks. The red dotted edges are the last crossing edges and the forbidden binding edges for the blocks.

First, consider a fixed planar embedding of C induced from the embedding of $H(C_i)$. Delete all the edges from $H(C_i)$ except for the edges on C_i and the edges on the boundary of each block B_j . For each block B_j , delete the edge between its leader and the last vertex (in the counterclockwise order). Finally also delete each edge (v_k, v_{k+1}) for each outer vertex v_k , which is a dominator of some block. Finally add the following edges for each dominator v_k . If v_k dominates only a single block B_j , then add the edge between v_k and the first vertex assigned to B_j , and the edge between v_{k+1} and the last vertex assigned to B_j . These two edges can be routed without a crossing near the (now removed) edge between the leader and the last vertex of B_j ; see Fig. 5. If v_k dominates more than one blocks $B_{j_1}, B_{j_2}, \dots, B_{j_t}$ in this cw-order, then we add the edge from v_k to the first vertex of B_{j_1} , and an edge from v_{k+1} to the last vertex of B_{j_t} . Also for $1 \leq l < t$, add an edge from the last vertex of B_{j_l} to the first vertex of $B_{j_{l+1}}$. Again all these edges can be routed near the (now removed) edges between the leader and the last vertex of the blocks. This gives a planar embedding of C .

We now show that all the edge assigned to page c_2 can be added in the interior of C without crossing. The edges assigned to c_2 are the first crossing edges of the blocks. For any block B , with leader d , its first crossing edge (if any) is between the first vertex u_1 assigned to B and the vertex x of B' preceding d , where B' is either the parent of B in \mathcal{T} or the sibling of B in \mathcal{T} just clockwise of it (Note that, in the later case, x is the last vertex of B'). We route such an edge as follows. We follow the boundary of B in cw-order from u_1 to d , then cross the boundary of B' if it is a sibling of B . Finally we follow the boundary of B' (counterclockwise in B' is the parent of B ; clockwise otherwise) to x ; see Fig. 5. The routed edges are planar and are in the interior of C . Hence they induce an outerplanar embedding, implying that edges assigned to c_2 can be embedded on a single page.

Finally, the edges assigned to page c_3, c_4 are the last crossing edges and the forbidden edges of the blocks. We show how we route them in the embedding of C without crossings. Consider a block B with the leader d and a last crossing edge e . Then e is between the last vertex of B and the vertex x on the parent of B in \mathcal{T} following d in the counterclockwise order. If B is at an even level in \mathcal{T} , we route the edge outside of C , following the edge to its dominator. Then we follow the boundary of C until we reach the last vertex of B' . Finally we follow the inside of the boundary of B' to x . If B is in the odd level, we route e inside following the boundary of B in the cw-order until d , then cross the boundary and finally follow the boundary of B' in the ccw-order to x ; see Fig. 5. For each block, if its last crossing edge follows its outside boundary, then the edges from its children blocks following its inside boundary and vice versa. Furthermore for the children of a block B in cw-order, their leaders also appear in the clockwise order on B and the edges from each child only covers the boundary of B only up to its leader. Thus these edge do not create crossing. Finally for a forbidden edge e of a block B , between it leader and its dominator, we route e in the same route for the last crossing edge; see Fig. 5. Thus all these edges along with C forms a planar graph with the Hamiltonian cycle C , and hence the can be embedded in the two pages c_3, c_4 . \square

3.2 Drawing 3-Connected 1-Planar Graphs

Here we describe a 12-page book embedding algorithm for any 3-connected 1-planar graph G . We first show how we order the vertices of G using the vertex placement order for 2-level subgraphs from the previous section. We then show how we assign the edges of G into a small number of pages.

As we described in the previous Section, we may assume that G is a normal planar-maximal 1-planar graph. We use a “peeling” technique to find a linear order for the vertices of the graph G level-by-level using the algorithm for Lemma 2. We first find and order of the vertices on the outer cycle C_0 (level 0 vertices) such that the vertices are placed in the cw-order around C_0 . We then traverse the graph outside in and iteratively use the algorithm for Lemma 2 to place the internal vertices. For the 2-level graphs between levels i and $i + 1$, we consider that the vertices of level i have already been placed and we place the vertices of level $i + 1$ using the algorithm for Lemma 2.

Consider a 2-level graph $H(C_i)$ between levels i and $i + 1$, where $C_i = \langle v_1, \dots v_t \rangle$ is the outer boundary of $H(C_i)$. If the cycle C_i is the first block in a 2-level graph between levels $i - 1$ and i , then the interval between the vertices of C_i does not contain any other vertex and we can use the algorithm in the previous section to place the level $i + 1$ vertices inside $H(C_i)$ between the already placed vertices of C_i . Otherwise there is some vertex of level $j < i$ between v_1 and v_2 , but the remaining vertices $(v_2, \dots v_t)$ are in a consecutive interval. In this case we again place the level $i + 1$ vertices inside $H(C_i)$ as in the algorithm for Lemma 2, but we place the vertices of level $i + 1$ blocks dominated by v_1 , just before v_2 (after all possible vertices of level $j < i$). In either case, the vertices on each level $i + 2$ cycles are placed in an interval with no vertices of level $j \leq i$ in between. Call this Algorithm **Order-Vertices**. We thus have the following lemma, whose proof follows from the above discussion; also see [19]:

Lemma 5. *Let Γ be the vertex order for a normal planar-maximal 1-planar graph G , obtained by Algorithm **Order-Vertices**. Let C_i be some level i cycle in G . Then all vertices at level $i + 1$ inside C_i are placed strictly between two consecutive level $i + 1$ vertices v_j and $v_{j'}$ in Γ .*

Lemma 5 implies that with this vertex order, no level $i + 1$ edge of G conflicts with any level j edge with $j < i$. We thus can iteratively use the drawing algorithm in Lemma 2 to obtain a book embedding of G as follows:

Theorem 1. *Every 3-connected 1-planar graph G has a book embedding on 14 pages.*

Proof. Let G be a normal planar-maximal 1-planar graph. Using Algorithm **Order-Vertices** we find a linear order of the vertices in G . We now again use the “peeling” technique to embed the edges of G level-by-level following the algorithm for Lemma 2. Let $p_1, \dots, p_6, c_1, \dots, c_8$ denote the 14 pages. We first embed the outer cycle C_0 (level 0 vertices) in a single page (page p_1). Then for each 2-level graph between levels i and $i + 1$, we iteratively use the pages p_1, p_2, p_3 , and c_1, c_2, c_3, c_4 to embed all the edges, when i is even; and we use the pages p_4, p_5, p_6 , and c_5, c_6, c_7, c_8 when i is odd. By Lemma 2, each 2-level subgraph is drawn without conflict, and by Lemma 5, the edge in any 2-level does not create conflict with any 2-level subgraph in a deeper level. \square

We can actually reduce the number of pages a little.

Theorem 2. *Every 3-connected 1-planar graph G has a book embedding on 12 pages.*

Proof. We can obtain a book embedding of G on 12 pages as a corollary of the construction in Theorem 1 after a post processing step. We note that all the 2-level planar graphs H' at all levels i of G , together induce a planar subgraph H of G , and are embedded on the six pages p_1, \dots, p_6 . Furthermore the order of the vertices in this book embedding is the same as the one obtained by the algorithm by Yannakakis [19] for a book embedding of H . Thus we use the algorithm by Yannakakis [19] to embed H on only four pages, resulting in a total of 12 pages. \square

4 Book Embedding of General 1-Planar Graphs

For the general case we may assume that the input graph is a planar-maximal graph and hence is 2-connected. We first extend the procedure of the *normalization* to the case of a planar-maximal 1-planar graph G . A pair of vertices $\{u, v\}$ of G share more than two crossing edge pairs if and only if $\{u, v\}$ form a separation pair in G [1]. During the normalization, for any separation pair $\{u, v\}$, we route the edge (u, v) such that all the crossing edge pairs with u, v as end-vertices falls on the same side of (u, v) ; see Fig. 6.

Suppose there is a separation pair $\{u, v\}$, with a decomposition $G - \{u, v\} = \{H_0, \dots, H_k\}$ for some $k \geq 1$. For any such component H_j , let H_j^* be the subgraph of G induced by the vertices of H_j and $\{u, v\}$. Then for at most one component H_j , u and v are not on the outerface of H_j^* . Assume thus without loss of generality that H_1^*, \dots, H_k^* all have u, v on the outerface. We call H_0 the *main* component and H_1, \dots, H_k the *inner components* for $\{u, v\}$. Also call H_1^*, \dots, H_k^* the *extended inner components*. The

edge (u, v) is called *separating edge*. Note that the inner components can be permuted and flipped at $\{u, v\}$. In a normalized planar maximal embedding $\mathcal{E}(G)$ of G , the inner components H_1, \dots, H_k are attached to (u, v) and are embedded on one side of (u, v) , say in this ccw-order at u . The components are separated by one or two pairs of crossing edges; see Fig. 6, and they may also be separated by copies of the separation edge [4,5]. The embeddings of the extended inner components are B - or W -configurations, defined by [17], and hence the boundaries of the inner components are triangles and quadrangles.

We now extend our 14-page book embedding of 3-connected 1-planar graphs and the “peeling technique” from Section 3.

Theorem 3. *Every 1-planar graph G has a book embedding on 16 pages.*

Proof. We proceed as in the case of 3-connected graphs. However we extend the peeling technique here to deal with the inner components for the separation pairs. Let the *main graph* G_0 be obtained from G by deleting all the inner components for all the separation pairs. Clearly G_0 is 3-connected. For each separation pair $\{u, v\}$, the edge (u, v) is a planar edge and if (u, v) is an edge of the main graph, then by the peeling technique, u, v are on the same level or on consecutive levels. Let H_1, \dots, H_k be the inner components for u, v . We then assign the vertices on the outer boundary O_j for each inner component H_j on the higher (i.e., deeper) of the two levels for u and v . For the remaining vertices of H_j we proceed with the peeling technique recursively and assign them to subsequent levels. Let u, v belong to some 2-level subgraph $H(C_i)$ of the main graph. Then the vertices on the outerboundary for each inner component for u, v and the edges between these outer vertices and u, v are on the 2-level subgraph for G . We now show how we place these vertices and assign the edges to augment the book embedding Γ of G_0 . In addition to the 14 pages used in Γ , we use two more pages q_1 and q_2 for 2-level subgraphs at odd and even levels, respectively.

For each separating edge (u, v) on the main graph, with u placed before v in Γ , insert the vertices on the outerboundary of each inner component for u, v consecutively, to the immediate left of v (in cw-order if v is on odd level and in ccw-order otherwise). If there is more than one inner component for u, v , the order of their placement is arbitrary. If several separating edges are incident to v , with the other end-vertex, say w_1, \dots, w_q , all placed before v and in this order in Γ , insert the vertices of the corresponding inner components in reverse order (i.e., the inner components for w_q, \dots , the inner components for w_1).

The edges on the outerboundary are assigned to c_1 or c_5 for odd and even levels, respectively; they do not create conflicts because they form simple cycles of length 3 or 4 and the vertices are consecutive. For each inner component H_j for separation pair

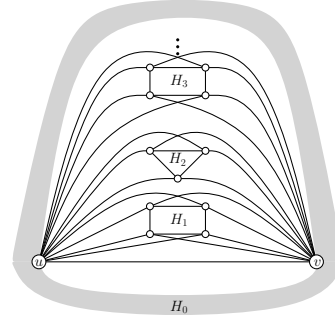


Fig. 6. A separation pair and the corresponding components.

$\{u, v\}$, the edges from u to the vertices on O_j are assigned to the same page as (u, v) , and the edges from v to the vertices of O_j are assigned to page q_1 (resp. q_2) for odd (resp. even) levels. Here the edges to v do not create conflicts with each other since they are all incident to v , and they do not create conflicts with other edges on q_1 (or q_2) since they are all placed immediately before v . Similarly the edges to u do not cross each other since they are all incident to u and they do not create conflicts with other edges in the same page since they follow the planar edge (u, v) assigned to the same page.

We recursively place the vertices inside each inner component during the computation for 2-level subgraphs on subsequent levels. Since we assign edges from 2-level subgraphs at odd and even levels on disjoint pages, following the argument of Lemma 5 the edges assigned to each of the 16 pages do not create conflicts. \square

It is *NP*-hard to determine whether a planar graph (which is a subclass of 1-planar graph) is sub-Hamiltonian. Hence, the minimum number of pages of a 1-planar graph cannot be computed efficiently. However, our algorithm takes only linear time, given a 1-planar embedding.

Theorem 4. *There is a linear time algorithm to construct book embedding of a general and a 3-connected 1-planar graph on 16 and 12 pages, respectively, given a 1-planar embedding.*

Proof. Given the 1-planar embedding, the normal planar maximal augmentation can be obtained in linear time. The crossing edges to be removed are selected in constant time per edge. Yannakakis algorithm for planar graphs runs in linear time, and the assignment of a removed edge to a page takes constant time per edge. Since there are at most $4n - 8$ edges, the algorithm runs in linear time. \square

If the input graph is planar and Hamiltonian, the order of the vertices from a Hamiltonian cycle induces a 2-page book embedding [3]. We can use this as follows.

Corollary 1. *A 1-planar graph G has a 4-page book embedding if the planar skeleton is Hamiltonian.*

Proof. Let $\mathcal{P}(G)$ be the planar skeleton of G with Hamiltonian cycle C . For each pair (a, b) and (c, d) of crossing edges assign (a, b) to a set X_1 and (c, d) to X_2 arbitrarily. By slight the abuse of notation, denote with X_1 (X_2) the subgraphs of G induced by X_1 (X_2). Both $G_1 = \mathcal{P}(G) \cup X_1$ and $G_2 = \mathcal{P}(G) \cup X_2$ contain Hamiltonian cycle C . Using the linear order of C we can embed G_1 in 2 pages and G_2 in 2-pages, yielding a book embedding for G on 4-pages with duplicate edges of $\mathcal{P}(G)$ removed. \square

5 Conclusion

We showed that general and 3-connected 1-planar graphs have a book embedding on 16 and 12 pages, respectively, and the book embedding can be computed in linear time from a given 1-planar embedding. Our bound improves upon the bound of 39 given by Bekos *et al.* [2]. The extended wheel graphs XW_{2k} for $k = 4, 5, 6$ require 4 pages; see Fig. 7. This was shown using a program which exhaustive searches all possible vertex orders and assignments of edges to pages. The natural open problem is to close the gap between the lower and upper bounds. Specifically, are there 1-planar graphs

that require even 5 pages? What is the lowest number of pages that suffices for 1-planar graphs, or 3-connected 1-planar graphs? These questions mirror the remaining big open problem for planar graphs: are there planar graphs that require 4 pages, or are all planar graphs embeddable on 3 pages?

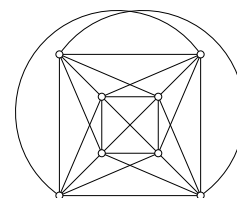


Fig. 7. The extended wheel graph XW_8 requires 4 pages.

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References

1. M. J. Alam, F. Brandenburg, and S. Kobourov. Straight-line grid drawings of 3-connected 1-planar graphs. In *Graph Drawing (GD'13)*, LNCS, pages 83–94, 2013.
2. M. A. Bekos, T. Bruckdorfer, M. Kaufmann, and C. N. Raftopoulou. 1-planar graphs have constant book thickness. In *23rd Annual European Symposium on Algorithms (ESA'15)*, volume 9294 of *Lecture Notes in Computer Science*, pages 130–141. Springer, 2015.
3. F. Bernhart and P. C. Kainen. The book thickness of a graph. *Journal of Combinatorial Theory, Series B*, 27(3):320–331, 1979.
4. F. J. Brandenburg. 1-visibility representations of 1-planar graphs. *Journal of Graph Algorithms and Applications*, 18(3):421–438, 2014.
5. F. J. Brandenburg. On 4-map graphs and 1-planar graphs and their recognition problem. *CoRR*, abs/1509.03447, 2015.
6. J. F. Buss and P. W. Shor. On the pagenumber of planar graphs. In *Proceedings of the 16th ACM Symposium on Theory of Computing*, pages 98–100, 1984.
7. Z.-Z. Chen, M. Grigni, and C. H. Papadimitriou. Recognizing hole-free 4-map graphs in cubic time. *Algorithmica*, 45(2):227–262, 2006.
8. F. Chung, F. Leighton, and A. Rosenberg. Embedding graphs in books: a layout problem with applications to VLSI design. *SIAM J. on Algebraic Discrete Methods*, 8(1):33–58, 1987.
9. V. Dujmovic and D. R. Wood. Graph treewidth and geometric thickness parameters. *Discrete & Computational Geometry*, 37(4):641–670, 2007.
10. A. Grigoriev and H. L. Bodlaender. Algorithms for graphs embeddable with few crossings per edge. *Algorithmica*, 49(1):1–11, 2007.
11. L. Heath. Embedding planar graphs in seven pages. In *25th Symposium on Foundations of Computer Science*, pages 74–89, 1984.
12. P. C. Kainen. Some recent results in topological graph theory. *Graphs and Combinatorics*, 406:76–108, 1974.
13. V. P. Korzhik and B. Mohar. Minimal obstructions for 1-immersions and hardness of 1-planarity testing. *Journal of Graph Theory*, 72(1):30–71, 2013.
14. L. T. Ollmann. On the book thicknesses of various graphs. In *4th Southeastern Conference on Combinatorics, Graph Theory and Computing*, volume 8, page 459, 1973.
15. J. Pach and G. Tóth. Graphs drawn with a few crossings per edge. *Combinatorica*, 17:427–439, 1997.
16. G. Ringel. Ein Sechsfarbenproblem auf der Kugel. *Abh. aus dem Math. Seminar der Univ. Hamburg*, 29:107–117, 1965.

17. C. Thomassen. Rectilinear drawings of graphs. *Journal of Graph Theory*, 12(3):335–341, 1988.
18. M. Yannakakis. Four pages are necessary and sufficient for planar graphs (extended abstract). In *ACM Symposium on Theory of Computing (STOC'86)*, pages 104–108, 1986.
19. M. Yannakakis. Embedding planar graphs in four pages. *Journal of Computer and System Sciences*, 38(1):36–67, 1989.